Local rings and localization

Def: A local ring is a ring R with only one maximal ideal m, sometimes denoted (R, m).

Note that every proper ideal in a local ring must be contained in m. Since the units u in a ring are exactly those s.t. $(u) = R$, we can also characterize local rings as follows.

Prop: R is a local ring $\Longleftrightarrow$ the set of nonunits is an ideal.

Pf: $(\Rightarrow)$ If (R, m) is a local ring, let $x \in R$.
If $x \in m$, then $(x) \neq R$ so x is not a unit.
If $x \notin m$, then $(x) \not\subseteq m$, so $(x) = R$, and thus x is a unit, so $m = \{\text{nonunits}\}$.

$(\Leftarrow)$ If $I \subseteq R$ is the set of nonunits of R, and is an ideal, let $M \subseteq R$ be a maximal ideal. M consists of nonunits, so $M \subseteq I$. Thus $M = I$. $\square$

Ex: All fields are local rings, w/ max'l ideal 0.

Often it is easier to work w/ local rings since the units

are exactly the elements not in the maximal ideal.
We can sometimes reduce to (or get closer to) the local case by "localizing" a ring, essentially by adding inverses for each elt outside of a given ideal.

Question: If R is a ring, for which elements can we add inverses?

If we add f^{-1} and g^{-1} then we're also adding $(fg)^{-1}$, so the set of elts U whose inverses we add must be <u>multiplicatively closed</u> (i.e. products of elts in U are in U including the "empty product" 1).

Ex:

1.) If $\overset{\neq 0}{t} \in R$, $\{1, t, t^2, \dots\}$ is multiplicatively closed.

2.) $P \subseteq R$ an ideal. $R - P$ is mult. closed iff P is prime.

3.) $R - \{0\}$ is mult. closed iff R is an integral domain.

Def: Let M be an R-module, $U \subseteq R$ multiplicatively closed. The <u>localization of M at U</u>, $U^{-1}M$, is the set of the equivalance classes of pairs $m \in M$, $u \in U$ (written $\frac{m}{u}$) w/ equivalence relation

$$\frac{m}{u} \sim \frac{m'}{u'} \iff \exists\, v \in U \text{ s.t. } vu'm = vum' \text{ in } M.$$

$U^{-1}M$ is an R-module by defining

$$\frac{m}{u} + \frac{m'}{u'} = \frac{u'm + um'}{uu'} \quad \text{and} \quad r\left(\frac{m}{u}\right) = \frac{rm}{u}.$$

In fact, $U^{-1}M$ is a $U^{-1}R$-module in the obvious way:

$$\left(\frac{r}{u}\right)\left(\frac{m}{u'}\right) = \frac{rm}{uu'}$$

Note: What happens if $v \in U$, $m \in M$ s.t. $vm = 0$?

Then $\frac{m}{1} = 0$.

In fact, the converse holds: If $\frac{m}{1} = 0$, $\exists\, v \in U$ s.t. $vm = 0$.

Ex: 1.) For an integral domain, $(R - \{0\})^{-1}R$ is the field of fractions, or <u>quotient ring</u> of R, denoted $K(R)$.

2.) More generally, if P is a prime ideal, and $u, v \notin P$, then $uv \notin P$, so $R - P$ is multiplicatively closed, and we denote

$R_P := (R-P)^{-1}R$. This is a local ring since the units are exactly the elts not in P.

If M is an R-module, then $M_p := (R-P)^{-1}M$ is an R_p-module.

Localization as a functor

If $\varphi: M \to N$ is a map of R-modules, and $U \subseteq R$ multiplicatively closed, there is a natural map

$$U^{-1}\varphi : U^{-1}M \to U^{-1}N \quad \text{s.t.} \quad \frac{m}{u} \mapsto \frac{\varphi(m)}{u}$$

of $U^{-1}R$-modules.

Check: $L \xrightarrow{\psi} M \xrightarrow{\varphi} N \implies U^{-1}(\varphi \circ \psi) = (U^{-1}\varphi) \circ (U^{-1}\psi)$, so localization is a functor from R-modules to $U^{-1}R$-modules.

Universal property

Suppose $\varphi: R \to S$ is a ring homomorphism, $U \subseteq R$ mult. closed.

Then as long as U gets sent to units in S, we can uniquely extend φ to

$$\varphi': U^{-1}R \to S \quad \text{by}$$

$$\frac{a}{b} \mapsto \varphi(a)\varphi(b)^{-1}.$$

This is the universal property of localization. i.e. if $\varphi: R \to S$ sends U to units in S,

$\exists$ unique φ' s.t. $\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \\ U^{-1}R \arrow[ur, dashed, "\varphi'"'] & \end{tikzcd}$ commutes.

Expansion and contraction of ideals

In order to describe the ideals in $U^{-1}R$, we first need some more terminology.

Let $\varphi: R \to S$ be a ring homomorphism.

Def:

1.) The <u>contraction</u> of an ideal $J \subseteq S$, denoted $J \cap R$, is the ideal $\varphi^{-1}(J) \subseteq R$.

2.) The <u>expansion</u> of an ideal $I \subseteq R$ to S, denoted IS, is the ideal of S generated by $\varphi(I)$.

Contraction and expansion are related as follows (see HW):

- $(J \cap R)S \subseteq J$ for all ideals $J \subseteq S$, and
- $I \subseteq (IS) \cap R$ for all ideals $I \subseteq R$.

Ideals of $U^{-1}R$

Let $\varphi: R \to U^{-1}R$ be the natural map.

If $I \subseteq U^{-1}R$, then $\frac{r}{u} \in I \Rightarrow r \in I$, so $r \in I \cap R$.

Thus, all numerators are in $R \cap I$

$$\Longrightarrow I = (U^{-1}R)(R \cap I).$$

$\Longrightarrow I \mapsto R \cap I$ is an injection from ideals of $U^{-1}R$ to ideals of R.

Which ideals of R are of the form $R \cap I$, where I is an ideal of $U^{-1}R$?

Ex: Let $R = k[x,y]$, $U = \{1, x, x^2, \dots\}$. Let $I = (x,y) \subseteq R$.

$\frac{x}{1}$ is a unit in $U^{-1}R$, so it's not contained in any proper ideal of $U^{-1}R$. Thus, $I \neq R \cap J$ for any ideal $J \subseteq U^{-1}R$.

More generally, if $I \subseteq R$ s.t. $I \cap U \neq \emptyset$, then I is not the contraction of any ideal in $U^{-1}R$. The converse doesn't hold:

Ex: $(xy) \cap \{1, x, x^2, \dots\} = \emptyset$.

However, set $J = (U^{-1}R)(xy) \subseteq U^{-1}R$.

Then $\frac{1}{x} \frac{xy}{1} = \frac{y}{1} \in J$.

so $y \in J \cap R$, but $y \notin (xy)$, so (xy) is not the contraction of any ideal.

However, it does hold for prime ideals:

Prop: The correspondence $P \longmapsto P \cap R$ is a bijection on prime ideals avoiding U.

Pf: First note that the preimage of a prime ideal is prime, so $P \cap R$ is indeed prime.

We've already showed it is injective, so we just need to show that if $Q \subseteq R$ is a prime, then

$$\frac{r}{1} \in (U^{-1}R)Q \Rightarrow r \in Q.$$

We leave the remainder as part of a HW problem. $\square$

Cor: If $P \subseteq R$ is prime, then the prime ideals of R_P are in one-to-one correspondence w/ Primes of R contained in P.

Rmk: Recall that the primes of R/I corr. to the primes in R that contain I.

Claim: If R is Noetherian, so is $U^{-1}R$.

Pf: If I is an ideal in $U^{-1}R$, then I is the expansion of some ideal J in R, which is f.g., so I must be as well. $\square$